A FORMULA IN THE THEORY OF SURFACES.

By R. D. Beetle.

The normal curvature $1/R$ and the geodesic torsion $1/T$ of a curve C lying on a surface S, whose first and second fundamental forms are $Edu^2 + 2Fdudv + Gdv^2$ and $Ddu^2 + 2D'dudv + D''dv^2$, are given by

$$\frac{1}{R} = \frac{Ddu^2 + 2D'dudv + D''dv^2}{Edu^2 + 2Fdudv + Gdv^2}, \tag{1}$$

and

$$\frac{1}{T} = \frac{(FD - ED')du^2 + (GD - ED'')dudv + (GD' - FD'')dv^2}{H(Edu^2 + 2Fdudv + Gdv^2)}, \tag{2}$$

where $H = \sqrt{EG - F^2}$. We may also write

$$\frac{1}{T} = \frac{1}{\tau} - \frac{d\omega}{ds}, \tag{3}$$

where $1/\tau$ is the torsion of the curve C, s is its arc, and ω is the angle which the normal to S makes with the principal normal of C.

In 1760, Euler* proved the relation

$$\frac{1}{R} = \frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) + \frac{1}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) \cos 2\theta, \tag{4}$$

where ρ_1 and ρ_2 are the principal radii of normal curvature, and θ is the angle between the directions whose radii of normal curvature are R and ρ_1. In 1848, Bonnet† derived the corresponding formula for $1/T$,

$$\frac{1}{T} = \frac{1}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) \sin 2\theta. \tag{5}$$

Introducing the total curvature

$$K = \frac{DD'' - D'^2}{H^2} = \frac{1}{\rho_1\rho_2}, \tag{6}$$

and the mean curvature

$$K_m = \frac{ED'' + GD - 2FD'}{H^2} = \frac{1}{\rho_1} + \frac{1}{\rho_2}, \tag{7}$$

we express (4) and (5) in the form

* L. Euler, Recherches sur la courbure des surfaces, Hist. de l'Acad. de Berlin, 16, 1760, p. 119.

† O. Bonnet, Memoire sur la theorie des surfaces, Journal de l'École Polytechnique, XXXII[e] Cahier, p. 1; 1848.

$$(8) \qquad \frac{1}{R} = \frac{1}{2}(K_m + \sqrt{K_m^2 - 4K} \cos 2\theta),$$

and

$$(9) \qquad \frac{1}{T} = \frac{1}{2} \sqrt{K_m^2 - 4K} \sin 2\theta.$$

Combining (8) and (9), we obtain the formula

$$(10) \qquad \frac{1}{R^2} + \frac{1}{T^2} = \frac{K_m}{R} - K,$$

which, so far as I know, has hitherto escaped notice. It is easily derived directly from (1), (2), (6) and (7) by using as parametric curves an orthogonal system with the curve C one of the curves $v =$ const.

The relation (10) can be given the following geometrical interpretation. For each point of S, the locus of the point whose Cartesian coördinates are $(1/R, 1/T)$ is a circle with center M at $(\frac{1}{2}K_m, 0)$ and radius $\frac{1}{2}\sqrt{K_m^2 - 4K}$. The circle meets the axis of $1/R$ in the points $A_1 = (1/\rho_1, 0)$ and $A_2 = (1/\rho_2, 0)$, corresponding to the principal directions on S. It meets the axis of $1/T$ in the points $B_1 = (0, \sqrt{-K})$ and $B_2 = (0, -\sqrt{-K})$, corresponding to the asymptotic directions on S. From (4) and (5), it follows that, for every point P on the circle, $\angle PMA_1 = 2\theta$. To orthogonal directions on the surface correspond, therefore, diametrically opposite points on the circle.

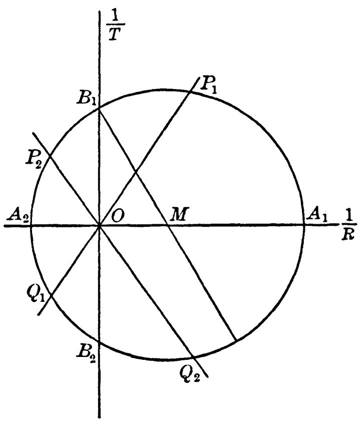

FIG. 1.

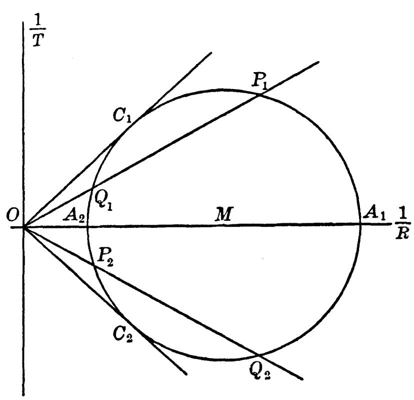

FIG. 2.

It is known, and easily verified directly from (1) and (2), that, if $(1/R_1, 1/T_1)$ and $(1/R_2, 1/T_2)$ correspond to conjugate directions, then $R_1/T_1 = -(R_2/T_2)$. It follows that, if P_1 and P_2 are the corresponding points on the circle, and O is the origin, the lines OP_1 and OP_2 are symmetrical with

respect to the coördinate axes. The lines OP_1 and OP_2 meet the circle in two other points, Q_1 and Q_2. The directions corresponding to P_1 and Q_2, or to P_2 and Q_1, are symmetrical with respect to the principal directions. The directions corresponding to Q_1 and Q_2 are conjugate. The directions corresponding to P_1 and Q_1, or to P_2 and Q_2, are in the relation called inverse-conjugate by Voss.*

Conversely, any two lines through O, and symmetrical with respect to the coördinate axes, meet the circle in four points which have the mutual relations just stated. To the points of contact,

$$C_1 = \left(\frac{2K}{K_m}, \frac{1}{K_m}\sqrt{K(K_m^2 - 4K)}\right),$$

$$C_2 = \left(\frac{2K}{K_m}, -\frac{1}{K_m}\sqrt{K(K_m^2 - 4K)}\right),$$

of the two tangents of the circle which pass through O, correspond the characteristic lines, the only self-inverse-conjugate lines.

From the geometric properties of the figure, we obtain the following theorems. With a few exceptions, they are well-known.

1. *On surfaces of negative total curvature (fig. 1), the asymptotic lines are real, but the characteristic lines are not. On surfaces of positive total curvature (fig. 2), the characteristic lines are real, but the asymptotic lines are not.*

2. *Each of the following is a characteristic property of minimal surfaces:*

(*a*) *the asymptotic lines form an orthogonal system;*

(*b*) *at each point, the value of $1/R^2 + 1/T^2$ is the same for all directions, being equal to minus the total curvature of the surface;*

(*c*) *at each point, the value of $1/\rho^2 + 1/\tau^2$ is the same for all the geodesics through the point, that is, the square of the total curvature of every geodesic is equal to the negative of the total curvature of the surface.*

For each is a necessary and sufficient condition that M fall on O.

3. *In order that a non-minimal surface be developable, it is necessary and sufficient that, at each point, the two asymptotic directions coincide.*

For the circle is then tangent at O to the axis of $1/T$.

4. *On developable minimal surfaces, that is, on planes, both asymptotic lines and lines of curvature are undetermined.*

The circle reduces to a point, the origin O.

5. *On spheres, the lines of curvature are undetermined, but the asymptotic lines are not.*

The circle reduces to the point M, distinct from O.

6. *The geodesic torsions in two orthogonal directions differ only in sign.*

* A. Voss, Über diejenigen Flächen, auf denen zwei Schaaren geodätischer Linien ein conjugirtes System bilden, Sitzungsberichte der K. Akademie zu München, vol. 18 (1888), p. 96.

For the corresponding points, being at the extremities of a diameter of the circle, have ordinates which differ only in sign.

7. *The sum of the normal curvatures in two orthogonal directions is constant at each point of* S, *and equals the mean curvature* K_m.

For the sum of the abscissas of two diametrically opposite points of the circle is equal to twice the abscissa of the center of the circle.

8. *It is a characteristic property of the orthogonal trajectories of the asymptotic lines that their normal curvature is equal to the mean curvature of the surface.*

9. *For two asymptotic lines through a point, the torsion at the point differs only in sign. The square of the torsion equals minus the total curvature of the surface.*

10. *For two characteristic lines through a point, the normal curvature at the point is the same. The radius of normal curvature in the direction of a characteristic line is equal to one-half the sum of the principal radii of normal curvature. The geodesic torsion of the characteristic lines through a point differs at the point only in sign.*

11. *The tangent of the angle between the asymptotic directions on any surface is*

$$\pm \frac{2\sqrt{-K}}{K_m} = \pm \frac{2\sqrt{-\rho_1\rho_2}}{\rho_1 + \rho_2}.$$

12. *The tangent of the angle between the characteristic directions on any surface is*

$$\pm 2\sqrt{\frac{K}{K_m{}^2 - 4K}} = \pm \frac{2\sqrt{\rho_1\rho_2}}{\rho_1 - \rho_2}.$$

13. *The geodesic torsion has its maximum and minimum values in the directions which bisect the principal directions.*

14. *For two inverse-conjugate directions, we have the relation* $R_1/T_1 = R_2/T_2$. *For two directions symmetric with respect to the lines of curvature, or for two conjugate directions, we have* $R_1/T_1 = -(R_2/T_2)$.

15. *On minimal surfaces, both for conjugate and inverse-conjugate directions, we have* $R_1 + R_2 = 0$. *In the first case* $T_1 = T_2$. *In the second case* $T_1 = -T_2$.

16. *For two conjugate or inverse-conjugate directions, the value of*

$$\left(\frac{1}{R_1{}^2} + \frac{1}{T_1{}^2}\right)\left(\frac{1}{R_2{}^2} + \frac{1}{T_2{}^2}\right)$$

is constant at a point, and equal to K^2.

17. *The square of the total curvature of a surface of Voss is equal to the product of the squares of the total curvatures of the conjugate geodesics.*

18. *For two conjugate or inverse-conjugate directions, the value of* $R_1 + R_2$ *is constant and equal to* $\rho_1 + \rho_2$.

This is an immediate consequence of 15, 16 and the relation (10).

19. *If* $R_1, R_2, \cdots, R_m$ *denote the values of* R *for* m *directions,* $m > 2$, *such that the angle between any two adjacent directions is* $2\pi/m$, *then*

$$\frac{1}{m}\left(\frac{1}{R_1}+\frac{1}{R_2}+\cdots+\frac{1}{R_m}\right)=\frac{1}{2}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right).$$

For the corresponding points are situated at equal intervals on the circumference of the circle, and, from elementary mechanical principles, their average distance from the axis of $1/T$ must be equal to the distance of the center of the circle from that axis. Since this is equally true for any other line in the plane of the circle, we can state the more general theorem.

20. *If* $R_1, \cdots, R_m, T_1, \cdots, T_m$ *denote the values of* R *and* T *for* m *directions,* $m > 2$, *such that the angle between any two adjacent directions is* $2\pi/m$, *then*

$$\frac{1}{m}\sum_{i=1}^{m}\left(\frac{a}{R_i}+\frac{b}{T_i}\right)=\frac{a}{2}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)$$

for any constant values of a *and* b.

In particular, we have

$$\left(\frac{1}{T_1}+\frac{1}{T_2}+\cdots+\frac{1}{T_m}\right)=0.$$

By considering more complicated properties of the figure, still more general theorems can be derived.* We have adopted, throughout, the point of view of the theory of surfaces, but it should be noted that the method employed is equally available for the study of conic sections.

PRINCETON, N. J.,
January, 1914.

* See, for example, M. Chasles, Diverses propriétés des rayons vecteurs et des diamétres d'une section conique. Propriétés analogues des rayons de courbure des sections normales d'une surface, en un point, Comptes Rendus, 26, 1848.